On the Nilpotent Multipliers of a Free Product with Topological Approach*

Zohreh Vasagh
Behrooz Mashayekhy†
Hanieh Mirebrahimi
Department of Pure Mathematics,
Center of Excellence in Analysis on Algebraic Structures,
Ferdowsi University of Mashhad,
P.O.Box 1159-91775, Mashhad, Iran.
E-mail: zo_va761@stu-mail.um.ac.ir, mashaf@math.um.ac.ir
& h_mirebrahimi@um.ac.ir

Abstract

In this paper, using the topological interpretation of the Baer invariant of a group G, $\mathcal{V}M(G)$, with respect to an arbitrary variety $\mathcal{V}$, we extend a result of Burns and Ellis (Math. Z. 226 (1997) 405-428) on second nilpotent multipliers of a free product of two groups to the c-nilpotent multipliers for all $c \geq 1$. In particular, we show that $M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H)$ when G and H are finite groups with some conditions or when G and H are two perfect groups.

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†Corresponding author

1 Introduction

Let $G \cong F/R$ be a free presentation of a group G, and $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then the Baer invariant of G with respect to $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) \cong \frac{R \bigcap V(F)}{[RV^*F]},$$

where $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \big\langle v(f_1, ..., f_{i-1}, f_i r, f_{i+1}, ..., f_n)\big(v(f_1, ..., f_n)\big)^{-1} |$$
$$r \in R, f_i \in F,\ 1 \leq i \leq n,\ v \in V,\ n \in \mathbb{N}\big\rangle.$$

Note that the Baer invariant of G is always abelian and independent of the presentation of G (see Leadham-Green and Mckay (1976)). In particular, if $\mathcal{V}$ is the variety of abelian groups, then the Baer invariant of G is the well-known notion, the Schur multiplier of G, which is isomorphic to the second homology group of G, $H_2(G, \mathbb{Z})$ (see Karpilovsky (1987)). If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, then the Baer invariant of the group G is called the c-nilpotent multiplier of G which is denoted by $M^{(c)}(G)$ (see Burns and Ellis (1997)).

Burns and Ellis (1997), using simplicial homotopy theory introduced a topological interpretation for the c-nilpotent multiplier of G and gave an interesting formula for the second nilpotent multiplier of the free product of two groups as follows:

$$\begin{aligned} M^{(2)}(G * H) \ &\cong M^{(2)}(G) \oplus M^{(2)}(H) \\ &\oplus \big(M(G) \otimes H^{ab}\big) \oplus \big(G^{ab} \otimes M(H)\big) \qquad (I) \\ &\oplus Tor(G^{ab}, H^{ab}). \end{aligned}$$

Also, Franco (1998) extended the above topological interpretation to the Bear invariant of a group G with respect to any variety $\mathcal{V}$. In this paper, first, we give a topological proof to show that the Bear invariant functor $\mathcal{V}M(G)$ commutes with the direct limits of a directed system of groups. Second, we intend to extend the formula (I) to the c-nilpotent multiplier of the free product of two groups for all $c \geq 1$. In particular, we show that $M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H)$, whenever one of the following conditions holds:

(i) G and H are finite abelian groups with coprime order.
(ii) G and H are finite groups with $\big(|G|, |H^{ab}|\big) = \big(|G^{ab}|, |H|\big) = 1$.
(iii) G and H are two finite groups with $\big(|G^{ab}|, |H^{ab}|\big) = \big(|M(G)|, |H|\big) = \big(|G^{ab}|, |M(H)|\big) = 1$.
(iv) G and H are two perfect groups.

2 Preliminaries and Notation

In this section, we recall some basic notations and properties of simplicial groups which will be needed in the sequel. We refer the reader to Curtis (1971) or Georss and Jardine (1999), for further details.

Definition 2.1. A *simplicial sets* $K_.$ is a sequence of sets $K_0, K_1, K_2, \ldots$ together with maps $d_i : K_n \to K_{n-1}$ (faces) and $s_i : K_n \to K_{n+1}$ (degeneracies), for each $0 \leq i \leq n$, with the following conditions:

$$\begin{array}{rcll} d_j d_i & = & d_{i-1} d_j & \text{for } j < i \\ s_j s_i & = & s_{i+1} s_j & \text{for } j \leq i \\ d_j s_i & = & \begin{cases} s_{i-1} d_j & \text{for } j < i; \\ identity & \text{for } j = i, i+1; \\ s_i d_{j-1} & \text{for } j > i+1. \end{cases} \end{array}$$

A *simplicial map* $f : K_. \to L_.$ means a sequence of functions $f_n : K_n \to L_n$, such that $f \circ d_i = d_i \circ f$, that is the following diagram commutes:

$$\begin{tikzcd} K_{n+1} \arrow[d, "f_{n+1}"'] & K_n \arrow[l, "s_i"'] \arrow[r, "d_i"] \arrow[d, "f_n"'] & K_{n-1} \arrow[d, "f_{n-1}"] \\ K_{n+1} & K_n \arrow[l, "s_i"'] \arrow[r, "d_i"] & K_{n-1}. \end{tikzcd}$$

Similar to topological spaces, the homotopy of two simplicial maps between simplicial sets and the homotopy groups of simplicial sets are defined. The category of simplicial sets and topological spaces can be related by two functors as follows:

- The *geometric realization*, $| - |$, is the functor from the category of simplicial sets to the category of CW complexes.

- The *singular simplicial*, $S_*(-)$, is the functor from the category of topological spaces to the category of simplicial sets.

A simplicial set $K_.$ is called a *simplicial group* if each K_i is a group and all faces and degeneracies are homomorphisms. There is a basic property of simplicial groups which due to Moore (1954-55), its homotopy groups $\pi_*(G_.)$ can be obtained as the homology of a certain chain complex (NG, ∂).

Definition 2.2. If $G_.$ is a simplicial group, then the *Moore complex* $(NG_., \partial)$ of $G_.$ is the (nonabelian) chain complex defined by $(NG)_n = \cap_{i=0}^{n-1} Kerd_i$ with $\partial_n : NG_n \to \ NG_{n-1}$ which is restriction of d_n.

A simplicial group $G_.$ is said to be *free* if each G_n is a free group and degeneracy homomorphisms s_i's send the free basis of G_n into the free basis for G_{n+1}.

Definition 2.3. For reduced simplicial set $K_.$ (i.e. $K_0 = *$), let $\mathbb{G}K$ be the simplicial group defined by $(\mathbb{G}K)_n$ which is the free group generated by $K_{n+1} \backslash s_0(K_n)$, and the face and degeneracy operators are the group homomorphisms such that

$$\begin{aligned} d_0^{\mathbb{G}K} k &= (d_1 k)(d_0 k)^{-1}, \\ d_i^{\mathbb{G}K} k &= d_{i+1} k \\ s_i^{\mathbb{G}K} k &= s_{i+1} k \end{aligned}$$

for $i > 0$ and $k \in K_{n+1}$. We can consider the above notion as a functor from reduced simplicial sets to free simplicial groups which is called *Kan's functor.*

In the following, we recall some results that will be needed in sequel.

Theorem 2.4. *(Curtis (1971)).*

(i) *For every simplicial group $G_.$ the homotopy group $\pi_n(G_.)$ is abelian even for $n = 1$.*

(ii) *Every epimorphism between simplicial groups is a fibration.*

(iii) *Let $G_.$ be a simplicial group, then $\pi_*(G_.) \cong H_*(NG_.)$.*

(iv) *For every simplicial set $K_.$, $\mathbb{G}K \simeq \Omega|K|$.*

3 Topological Approach to Baer Invariants

Let $X = K(G,1)$ be the Eilenberg-MacLane space of G. Then Burns and Ellis (1997) presented an isomorphism $M^{(c)}(G) \cong \pi_1\big(K_./\gamma_{c+1}(K_.)\big)$, where $K_.$ is the free simplicial group obtained from X by applying Kan's functor to the reduced singular simplicial set of X. Burns and Ellis's interpretation for $c = 1$ is $M(G) \cong \pi_1\big(K_./\gamma_2(K_.)\big)$. Also, Kan (1958) proved that $\pi_*\big(\mathbb{G}X/\gamma_2(\mathbb{G}X)\big) \cong H_{*+1}(X)$, where $\mathbb{G}$ is the Kan's functor. Hence $H_2(G) \cong R \cap F/[R,F] = M(G)$ which is the Hopf's formula, where $G = F/R$ is a free presentation for G.

Using the above notions and similar to the Burns and Ellis's interpretation we can give a topological interpretation for the Baer invariant of a group G with respect to any variety $\mathcal{V}$. We recall that the following theorem was proved categorically by Franco (1998),.

Theorem 3.1. *Let $X = K(G,1)$ be the Eilenberg-MacLane space of G and $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then the following isomorphisms hold.*

$$\begin{aligned} \pi_1\big(K_./V(K_.)\big) &\cong \mathcal{V}M(G) \\ \pi_0\big(K_./V(K_.)\big) &\cong G/V(G), \end{aligned}$$

where $K_.$ is the free simplicial group obtained from X by applying Kan's functor to the reduced singular simplicial set of X.

Proof. Let $G \cong F/R$ be a free presentation of G. Then for the simplicial group $K_.$ obtained by applying Kan's functor to the reduced of $S_*(X)$, we have $|K_.| \simeq \Omega X$ (see Wu (2007)). Therefore $(K_.)_0 = F$ and $(K_.)_1 = R \rtimes F$ and $d_0^1(r,f) = f$ and $d_1^1(r,f) = rf$ (see Burns and Ellis (1997)). Hence $\big(K_./V(K_.)\big)_0 = F/V(F)$, $\big(K_./V(K_.)\big)_1 = R/[RV^*F] \rtimes F/V(F)$ and $\bar{d}_0^1$ and $\bar{d}_1^1$ are induced by d_1^0 and d_1^1, respectively. We consider the Moore chain complexes $N\big(K_./V(K_.)\big)$ and $N\big(V(K_.)\big)$. By Theorem 2.4 (iii), we have $\pi_0\big(K_./V(K_.)\big) \cong G/V(G)$ and $\pi_0\big(V(K_.)\big) \cong V(F)/[RV^*F]$. By Theorem 2.4 (ii), the following exact sequence of simplicial groups is a fibration.

$$0 \to V(K_.) \to K_. \to \frac{K_.}{V(K_.)} \to 0.$$

Thus it induces the long exact sequence in homotopy groups as follows:

$$\cdots \to \pi_1(K_.) \to \pi_1\big(\frac{K_.}{V(K_.)}\big) \to \pi_0\big(V(K_.)\big) \overset{\pi_0(\subseteq)}{\to} \pi_0(K_.) \to \pi_0\big(\frac{K_.}{V(K_.)}\big) \to 0.$$

Also $\pi_1(K_.) \cong \pi_1(\Omega X) \cong \pi_2(X) = 0$ and similarly $\pi_0(K_.) \cong \pi_1(X) \cong G$. Hence $\pi_1\big(K_./V(K_.)\big) \cong \ker\big(\pi_0(\subseteq)\big) \cong V(F) \cap R/[RV^*F]$. $\square$

Using the above topological interpretation of Baer invariants, we intend to study the behavior of Baer invariants with direct limits with topological approach. First, we need to find the behavior of homotopy groups of simplicial groups with respect to the direct limit.

Theorem 3.2. *Let $\{{}^jG_., \varphi_i^j; i, j \in J\}$ be a direct system of simplicial groups $\{{}^jG_.\}$ indexed by a directed set J. Then*

$$\pi_n\big(\varinjlim_{j\in J} {}^jG_.\big) \cong \varinjlim_{j\in J} \pi_n\big({}^jG_.\big).$$

Proof. Let ${}^jd_i^k : {}^jG_k \to {}^jG_{k+1}$ and ${}^js_i^k : {}^jG_k \to {}^jG_{k-1}$ be faces and degeneracies, for $0 \le i \le k$. Recall that the direct limit of simplicial groups can be considered as a simplicial group as follows

$$\begin{aligned} (\varinjlim_{j\in J} {}^jG_.)_n &= \varinjlim_{j\in J} ({}^jG_.)_n \\ d_i^n &= \varinjlim_{j\in J} ({}^jd_i^n) \\ s_i^n &= \varinjlim_{j\in J} ({}^js_i^n). \end{aligned}$$

We have the following commutative diagram

$$\begin{tikzcd}
\varinjlim_{j\in J} ({}^jG_.)_{n+1} & \varinjlim_{j\in J} ({}^jG_.)_n \arrow[l, "\varinjlim_{j\in J}({}^js_i^n)"'] \arrow[r, "\varinjlim_{j\in J}({}^jd_i^n)"] & \varinjlim_{j\in J} ({}^jG_.)_{n+1} \\
{}^kG_{n+1} \arrow[u, "({}^k\theta)_{n+1}"] & {}^kG_n \arrow[l, "{}^ks_i^n"'] \arrow[r, "{}^kd_i^n"] \arrow[u, "({}^k\theta)_n"] & {}^kG_{n-1} \arrow[u, "({}^k\theta)_{n-1}"] \\
{}^lG_{n+1} \arrow[u, "(\varphi_l^k)_{n+1}"] & {}^lG_n \arrow[l, "{}^ls_i^n"'] \arrow[r, "{}^ld_i^n"] \arrow[u, "(\varphi_l^k)_n"] & {}^lG_{n-1}. \arrow[u, "(\varphi_l^k)_{n-1}"]
\end{tikzcd} \tag{3.1}$$

Consider the Moore chain complex $N(\varinjlim_{j\in J} {}^jG_.)$ as follows:

$$\cdots \xrightarrow{\varinjlim {}^jd_3^3} \ker \varinjlim_{j\in J} {}^jd_0^2 \cap \ker \varinjlim_{j\in J} {}^jd_1^2 \xrightarrow{\varinjlim {}^jd_2^2} \ker \varinjlim_{j\in J} {}^jd_0^1 \xrightarrow{\varinjlim {}^jd_1^1} \varinjlim_{j\in J} ({}^jG_.)_0.$$

Since direct limit of a directed system preserves exact sequence and

$$\varinjlim_{j\in J}(\ker {}^j d^i_k) \cap \varinjlim_{j\in J}(\ker {}^j d^i_{k'}) = \varinjlim_{j\in J}(\ker {}^j d^i_k \cap \ker {}^j d^i_{k'}),$$

we have the following chain complex

$$\cdots \xrightarrow{\varinjlim {}^j d^3_3} \varinjlim_{j\in J}(\ker \quad {}^j d^2_0 \cap \ker \quad {}^j d^2_1) \xrightarrow{\varinjlim {}^j d^2_2} \varinjlim_{j\in J} \ker {}^j d^1_0 \xrightarrow{\varinjlim {}^j d^1_1} \varinjlim_{j\in J} ({}^j G_.)_0.$$

Hence $N(\varinjlim_{j\in J} {}^j G_.) \cong \varinjlim_{j\in J} N({}^j G_.)$ when J is a directed set. Also, homology functor preserves direct limit of directed system of simplicial groups. Therefore, using Theorem 2.4 (iii), we have

$$\pi_n(\varinjlim_{j\in J} {}^j G_.) \cong H_n\big(N(\varinjlim_{j\in J} {}^j G_.)\big) \cong \varinjlim_{j\in J} H_n\big(N({}^j G_.)\big) \cong \varinjlim_{j\in J} \pi_n({}^j G_.).$$

$\square$

Remark 3.3. Note that homotopy groups do not commute with direct limits of topological spaces in general and hence Theorem 3.2 does not hold in the category of topological spaces. To prove this, Goodwillie (2004) gives the following intresting example.

Let $S^1 = \{(x,y) \in \mathbb{R} | x^2 + y^2 = 1\}$ be the unit circle. Let $A_n = \{(x,y) \in \mathbb{R} | x^2 + y^2 = 1, x \leq 1 - 1/n\}$ be a sequence of closed arcs in S^1 such that A_n is in the interior of A_{n+1} and such that the union U of all the A_n is the complement of a point in S^1. Let X_n be S^1/A_n. The direct limit of the diagram of circles $X_1 \to X_2 \to \ldots$ is S^1/U, a two-point space in which only one of the points is closed. Or if one prefers to form the colimit in the category of Hausdorff or T_1 spaces, then the colimit is a point. Either way, π_1 dose not commute with the direct limit.

Note that homotopy groups preserve the direct limit of a filtered based spaces (for more details see May (1999)).

Now we are in position to give a topological proof for the following theorem which was proved algebraically by Moghaddam (1980).

Theorem 3.4. *Let $\{G_i, \varphi^j_i, i \in I\}$ be a directed system of groups, then*

$$\varinjlim_{i\in I} \mathcal{V}M(G_i) \cong \mathcal{V}M\big(\varinjlim_{i\in I}(G_i)\big).$$

Proof. Let K_i be a free simplicial group corresponding to G_i. By Lemma 3.2 of Moghaddam (1980), $\varinjlim_{i\in I} K_i$ is the free simplicial group and Theorem (3.2) implies that $\varinjlim_{i\in I} K_i$ is a free simplicial resolution corresponding to $\varinjlim_{i\in I} G_i$. Hence we have

$$\begin{aligned}\mathcal{V}M\big(\varinjlim_{i\in I}(G_i)\big) &\cong \pi_1\Big(\frac{\varinjlim K_i}{V\big(\varinjlim K_i\big)}\Big) \cong \pi_1 \varinjlim \Big(\frac{K_i}{V\big(K_i\big)}\Big)\\ &\cong \varinjlim \pi_1\Big(\frac{K_i}{V\big(K_i\big)}\Big) \cong \varinjlim_{i\in I} \mathcal{V}M(G_i).\end{aligned}$$

□

4 Main Results

In this section, by considering the variety of nilpotent groups, we intend to compute the nilpotent multipliers of the free product of two groups.

Proposition 4.1. *Let $F = K * L$ be the free product of two free groups K and L and let $\varphi : F \to K \times L$ be the natural epimorphism. Then for all $c \geq 1$, there exists the following short exact sequence*

$$0 \to ker\bar{\varphi}_c \to \frac{F}{\gamma_{c+1}(F)} \xrightarrow{\bar{\varphi}_c} \frac{K}{\gamma_{c+1}(K)} \times \frac{L}{\gamma_{c+1}(L)} \to 0,$$

where $\ker\bar{\varphi}_c \cong \frac{[K,L]^F}{[K,L,_{c-1}F]^F}$ *which satisfies in the following exact sequence*

$$0 \to \frac{[K,L,_{c-2}F]^F}{[K,L,_{c-1}F]^F} \to ker\bar{\varphi}_c \to \frac{[K,L]^F}{[K,L,_{c-2}F]^F} \to 0.$$

Moreover, we have the following isomorphism

$$\frac{[K,L,\ _{c-2}F]^F}{[K,L,\ _{c-1}F]^F} \cong \oplus \sum_{for\ \ some\ \ i+j=c} \underbrace{K^{ab}\otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab}\otimes ... \otimes L^{ab}}_{j-times}.$$

Proof. Clearly the natural epimorphism $\varphi : F \to K \times L$ induces an epimorphism

$$\bar{\varphi}_c : F\ /\gamma_{c+1}(F) \to K/\gamma_{c+1}(K) \times L/\gamma_{c+1}(L)$$

given by

$$\bar{\varphi}_c\big(\omega\gamma_{c+1}(F)\big) = \big(\omega_1\gamma_{c+1}(K), \omega_2\gamma_{c+1}(L)\big),$$

for all $c \geq 1$, where $\varphi(\omega) = \big(\omega_1, \omega_2\big)$. Therefore, we have

$$ker\bar{\varphi}_c \cong \frac{[K,L]^F\gamma_{c+1}(F)}{\gamma_{c+1}(F)} \cong \frac{[K,L]^F}{[K,L]^F \cap \gamma_{c+1}(F)} \cong \frac{[K,L]^F}{[K,L,\ _{c-1}F]^F}.$$

Hence the following exact sequence exists

$$0 \to \frac{[K,L,\ _{c-2}F]^F}{[K,L,\ _{c-1}F]^F} \to ker\bar{\varphi}_c \to \frac{[K,L]^F}{[K,L,\ _{c-2}F]^F} \to 0.$$

Moreover, let K and L be free groups freely generated by $\{x_1, \cdots, x_m\}$ and $\{x_{m+1}, \cdots, x_{m+n}\}$, respectively. Then by a theorem of Hall (1959), it is easy to show that $[K,L,\ _{c-2}F]^F \ / \ [K,L,\ _{c-1}F]^F$ is a free abelian group with the basis $\bar{B} = \{b[K,L,\ _{c-1}F]^F | b \in B\}$, where $B = B_1 - B_2 - B_3$ in which B_1, B_2, B_3 are the set of all basic commutators of weight c on $\{x_1, \cdots, x_m, \cdots, x_{m+n}\}$, $\{x_1, \cdots, x_m\}$ and $\{x_{m+1}, \cdots, x_{m+n}\}$, respectively. Now by universal property of free abelian groups and tensor products we have the following isomorphism:

$$\frac{[K,L,\ _{c-2}F]^F}{[K,L,\ _{c-1}F]^F} \cong \oplus \sum_{for\ some\ i+j=c} \underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{j-times}.$$

Note that the number of copies in the above direct sum is the number of all basic commutators subgroups of weight c on K and L. $\square$

Theorem 4.2. *Let G, H be two groups with*

$$G^{ab} \otimes H^{ab} = M^{(1)}(G) \otimes H^{ab} = M^{(1)}(H) \otimes G^{ab} = Tor(G^{ab}, H^{ab}) = 0.$$

Then the following isomorphism holds, for all $c \geq 1$,

$$M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H).$$

Proof. For $c = 1$, by a well-known result on Schur multiplier of the free product (see Karpilovsky (1987)), we have the following isomorphism:

$$M^{(1)}(G * H) \cong M^{(1)}(G) \oplus M^{(1)}(H).$$

Now we discuss in more details on the Burns and Ellis's method (1997), and extend the method to any $c \geq 2$. Let $K_.$ and $L_.$ be free simplicial groups corresponding to $K(G,1)$ and $K(H,1)$, respectively. By van-Kampen theorem we have $X \vee Y \cong K(G*H,1)$ so that the free simplicial group $F_.$ which obtained by applying Kan's functor to the reduced singular simplicial set of $X \vee Y$ is equal to $K_. * L_.$. Therefore $M^{(c)}(G* H) \cong \pi_1\big(F_./\gamma_{c+1}(F_.)\big)$. By Proposition 4.1, consider the following short exact sequence of simplicial groups

$$0 \to (\ker \bar{\varphi}_c)_. \to \frac{F_.}{\gamma_{c+1}(F_.)} \xrightarrow{\bar{\varphi}_c} \frac{K_.}{\gamma_{c+1}(K_.)} \times \frac{L_.}{\gamma_{c+1}(L_.)} \to 0,$$

where $(ker\bar{\varphi}_c)_.$ is a simplicial group defined by $\big((ker\bar{\varphi}_c)_.\big)_n = ker(\bar{\varphi}_c)_n$. Theorem 2.4 (ii) yields the following long exact sequence

$$\begin{aligned}\cdots \quad &\to \pi_2\big((\ker \bar{\varphi}_c)_.\big) \to \pi_2\big(\tfrac{F_.}{\gamma_{c+1}(F_.)}\big) \overset{\pi_2(\bar{\varphi}_c)}{\to} \pi_2\big(\tfrac{K_.}{\gamma_{c+1}(K_.)}\big) \oplus \pi_2\big(\tfrac{L_.}{\gamma_{c+1}(L_.)}\big)\\ &\to \pi_1\big((\ker \bar{\varphi}_c)_.\big) \to \pi_1\big(\tfrac{F_.}{\gamma_{c+1}(F_.)}\big) \overset{\pi_1(\bar{\varphi}_c)}{\to} \pi_1\big(\tfrac{K_.}{\gamma_{c+1}(K_.)}\big) \oplus \pi_1\big(\tfrac{L_.}{\gamma_{c+1}(L_.)}\big) \to \cdots .\end{aligned}$$

Let $\alpha_n^K : \pi_n\big(F_./\gamma_{c+1}(F_.)\big) \to \pi_n\big(K_./\gamma_{c+1}(K_.)\big)$ and $\alpha_n^L : \pi_n\big(F_./\gamma_{c+1}(F_.)\big) \to \pi_n\big(L_./\gamma_{c+1}(L_.)\big)$ be homomorphisms induced by continuous maps from $X \vee Y$ to X and Y, respectively. Since $\pi_n\big(K_./\gamma_{c+1}(K_.)\big) \oplus \pi_n\big(L_./\gamma_{c+1}(L_.)\big)$ is a coproduct in the category of groups so there exists a unique homomorphism $\psi_n : \pi_n(K_./\gamma_{c+1}(K_.)) \oplus \pi_n\big(L_./\gamma_{c+1}(L_.)\big) \to \pi_n\big(F_./\gamma_{c+1}(F_.)\big)$ such that $p_n^K o\psi_n = \alpha_n^K$ and $p_n^L o\psi_n = \alpha_n^L$, where p_n^L and p_n^K are projection maps. Therefore $\psi_n o\pi_n(\bar{\varphi}_c) = id$ and consequently

$$\pi_1(ker\bar{\varphi}_c) \oplus \pi_1\big(K_./\gamma_{c+1}(K_.)\big) \oplus \pi_1\big(L_./\gamma_{c+1}(L_.)\big) \cong \pi_1\big(F_./\gamma_{c+1}(F_.)\big).$$

By Proposition 4.1, we have the following exact sequence of simplicial groups

$$0 \to \frac{[K, L,_{c-2} F]^F}{[K, L,_{c-1} F]^F} \to ker\bar{\varphi}_c \to \frac{[K, L]^F}{[K, L,_{c-2} F]^F} \to 0.$$

Theorem 2.4 (ii) yields the following long exact sequence of homotopy groups which in low dimension takes the following form:

$$\cdots \to \pi_1(\frac{[K, L,\ _{c-2}F]^F}{[K, L\ ,_{c-1} F]^F}) \to \pi_1(ker\bar{\varphi}_c) \to \pi_1(ker\bar{\varphi}_{c-1}) \to \cdots .$$

By induction on c, we prove that $\pi_1(ker\bar{\varphi}_c)=0$. For $c=2$, Burns and Ellis (1997) proved that $ker\bar{\varphi}_2 \cong K^{ab}_{.} \otimes L^{ab}_{.}$. Hence

$$\begin{array}{ll}\pi_1(ker\bar{\varphi}_2) & \cong H_1\big(N(K^{ab}_{.} \otimes L^{ab}_{.})\big) \\ & \cong H_1\big(N(K^{ab}_{.}) \otimes N(L^{ab}_{.})\big) \\ & \cong H_1\big(N(K^{ab}_{.})\big) \otimes H_0\big(N(L^{ab}_{.})\big) \oplus H_0\big(N(K^{ab}_{.})\big) \otimes H_1\big(N(L^{ab}_{.})\big) \\ & \oplus Tor\Big(H_0\big(N(K^{ab}_{.})\big), H_0\big(N(L^{ab}_{.})\big)\Big) \\ & \cong M^{(1)}(G) \otimes H^{ab} \oplus M^{(1)}(H) \otimes G^{ab} \oplus Tor(G^{ab}, H^{ab}).\end{array}$$

Similarly, we can prove that

$$\pi_0(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{j-times}) \cong \underbrace{G^{ab} \otimes ... \otimes G^{ab}}_{i-times} \otimes \underbrace{H^{ab} \otimes ... \otimes H^{ab}}_{j-times}.$$

Now let $\pi_1(ker\bar{\varphi}_{c-1})=0$. We are going to show that $\pi_1(ker\bar{\varphi}_c)=0$. Since

$$\frac{[K, L, \ _{c-2}F]^F}{[K, L, \ _{c-1}F]^F} \cong \oplus \sum_{for\ \ some\ \ i+j=c} \underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... L^{ab}}_{j-times},$$

it is enough to compute $\pi_1(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{j-times})$. Since $i, j \neq 0$, we have

$$\begin{array}{l}\pi_1(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{i-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{j-times}) \\ \cong \pi_1(K^{ab} \otimes L^{ab}) \otimes \pi_0(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{(i-1)-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{(j-1)-times}) \\ \oplus \pi_0(K^{ab} \otimes L^{ab}) \otimes \pi_1(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{(i-1)-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{(j-1)-times}) \\ \oplus Tor\big(\pi_0(K^{ab} \otimes L^{ab}), \pi_0(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{(i-1)-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{(j-1)-times})\big).\end{array}$$

By the hypothesis, we have $\pi_1(\underbrace{K^{ab} \otimes ... \otimes K^{ab}}_{(i-1)-times} \otimes \underbrace{L^{ab} \otimes ... \otimes L^{ab}}_{(j-1)-times})=0$. Hence $\pi_1(ker\bar{\varphi}_c)=\ 0$. □

Corollary 4.3. *Let G and H be two groups. Then, for all $c \geq 1$, we have the following isomorphism*

$$M^{(c)}(G * H) \cong M^{(c)}(G) \oplus M^{(c)}(H),$$

if one of the following conditions holds:
(i) *G and H are two abelian groups with coprime orders.*
(ii) *G and H are two finite groups with $\big(|G|, |H^{ab}|\big) = \big(|G^{ab}|, |H|\big) = 1$.*
(iii) *G and H are two finite groups with*

$$\big(|G^{ab}|, |H^{ab}|\big) = \big(|M(G)|, |H|\big) = \big(|G^{ab}|, |M(H)|\big) = 1.$$

(vi) *G and H are two perfect groups.*

Note that parts $(i) - (iii)$ of the above corollary are vast generalizations of a result of the second author (see Mashayekhy (2002)).

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